JOURNAL OF AIRCRAFT
Vol. 30, No. 2, March–April 1993

Structural Analysis of Box Beams Using Symbolic Manipulation Technique

M. Sathyamoorthy* and Ravindra Sirigiri†
Clarkson University, Potsdam, New York 13699

The aeroelastic analysis of aircraft wings requires an accurate determination of the influence coefficients. In the past, energy methods have been commonly used to analyze box-type structures and the results have been found to agree well with the experiments. However, when analysis of large wing-type structures is desired, it becomes necessary to automate the energy method. In this article, a method has been developed based on symbolic manipulation as an automated technique to find solutions to box-type structures. Various manipulations required for the energy method have been automatically implemented in a computer program with solutions available at each stage in a symbolic form. The numerical results for several example problems have been compared with alternate theoretical as well as experimental results. Good agreement has been noted in all the cases considered in this article.

Introduction

MATRIX methods have long been used for the structural analysis of complex structures. With the availability of high-speed computers, it is now possible to use the matrix-based finite-element method in the analysis and design of very complex structures. In particular, the finite-element method is being extensively used as an analysis tool to verify test data on many practical engineering applications. Also, as a result of enormous effort in this area, several efficient finite-element codes have been developed to meet the ever-increasing demand for accurate theoretical modeling, analysis, and results. A review of the finite-element method in connection with the analysis of highly indeterminate complex structures is presented in Ref. 1.

For wing-type structures, conventional methods of analysis using the simple bending theory of beams do not yield good results. To overcome this difficulty, the energy method has been commonly used in the past.[2] This energy method is based primarily on equilibrium considerations with the use of elastic strain energy in the structure for deriving the displacements. In the case of highly indeterminate structures such as aircraft wings, the equilibrium conditions alone are not sufficient to solve for the displacements. Therefore, the condition that the actual displacements and the stress distribution correspond to the minimum of strain energy is used in addition to the equilibrium conditions to get the displacement and to further evaluate the stresses.

A review of the literature indicates that in addition to the energy method Klein's method has been used for the analysis of aircraft wing structures. This method is based on the solution of a set of equations consisting of equilibrium and force-displacement relations of all the elements of the wing. This method was developed by Klein[3] and was used to analyze wings subjected to bending-type loadings. In the original development of the method for semimonocoque structures, it was assumed that the wing structure was composed of uniform, nontapered axial elements (spar, rib, and web chords) taking only axial loads and constant thickness sheets (cover panels, spar, and rib webs) taking only shear loads in the form of constant shear flows.

Although Klein's method could be modified to account for the variable nature of spars, ribs, webs, and panels that actually exist in real-wing structures, it is thought that the results may still not be very accurate. This article is, therefore, concerned with the development of an alternate method for the deflection analysis of complicated wing structures. To examine the applicability of the symbolic manipulation technique, several cases of box-beams are considered. For each of the cases considered, the influence coefficients are determined by applying a load at each joint. For all the cases considered here the equilibrium equations are automatically generated and combined with the minimization of the energy method to obtain the required solutions. Manipulation of the equilibrium equations, minimization of the strain energy, as well as the final solution of the resulting equations, are all accomplished in a systematic manner using a symbolic manipulation procedure. For the cases where there are a large number of boxes put together, the equilibrium equations generated here have been verified using a program originally developed at the U.S. Army Ballistic Research Laboratory.[4] This program has recently been thoroughly reviewed and revised to generate the exact number of the required equations for complex indeterminate structures.[5] In all the example problems considered in this article, the results generated by the symbolic manipulation technique are found to have good agreement with the available test results as well as alternate theoretical results found in the literature. For all the cases considered so far, the spars, ribs, panels, and webs are assumed to have constant cross-sectional areas and thicknesses. In a later study, it is proposed to extend this method for the analysis of complicated aircraft wing problems with variable thickness plate elements, and spars and ribs with continuously varying cross-sectional areas.

MACSYMA and the Symbolic Manipulation Technique

The idea of performing algebraic manipulations on computers is not new. Since 1960, numerous programs have appeared in the market to show that tedious mathematical manipulations can be easily carried out using computers. This is a remarkable departure from the numerical areas usually attributed to computers. The language LISP, dates from this period and opened the way to the first spectacular demonstrations of several possibilities, including formal integration and proof of theorems. The availability of time-sharing sys-

Received March 20, 1991; revision received Jan. 24, 1992; accepted for publication Feb. 15, 1992.

*Executive Officer and Associate Professor, Department of Mechanical and Aeronautical Engineering.
†Systems Engineer, Micro Equipment Corp., Norcross, GA.

tems have contributed a great deal to the generalization of programs for algebraic manipulations.

Most symbolic analyses in structural mechanics areas have so far been carried out manually using systematic application of standard analytical procedures. However, in recent years, programs that perform symbolic manipulations, such as MAC's SYmboic MAnipulation system (MACSYMA), are being used by researchers to get a deeper understanding of a problem, which is usually difficult to see from strictly numerical solutions.

MACSYMA[6] is a large computer programming system written in LISP commonly used to perform symbolic, as well as numerical, mathematical computations. MACSYMA can be conveniently used to do various mathematical operations such as differentiation, integration, taking limits, solving systems of linear and nonlinear sets of equations, factorizing polynomials, expanding functions in series, solving differential equations, plotting curves, and manipulating matrices and tensors. All of these operations can be done by means of a series of input commands and the output is usually available in a symbolic form.

In terms of using MACSYMA, the current research involves mainly four different mathematical operations in order to get solutions to the problems of box-type structures. They are matrix inversion, integration, differentiation of the energy expression with respect to the redundant as well as the applied forces, and the solution of the resulting set of simultaneous equations. Although all of these operations can be carried out manually, it is impossible to apply them to the solution of large-size problems such as multiple-bay box beams and wings. The laborious mathematical operations associated with the solution of such large-size problems can, however, be performed conveniently and in a very systematic manner using MACSYMA. The purpose of this article, therefore, is to use MACSYMA and develop a new technique called symbolic manipulation technique (SMT).

Example Problems

In this section, the details of the SMT are presented considering an example of a single-bay box beam shown in Fig. 1. The box beam has the dimensions "a" from tip to root, "b" from left (leading edge) to right (trailing edge), and "h" from top to bottom. The areas of cross section of all spars and ribs are equal to A_f and the thicknesses of all cover panels and webs are assumed to be uniform. If the displacements at A and B are assumed to be zero, then the total number of unknown displacements are six corresponding to three displacement components at each of the two nodes 1 and 2. The number of unknown forces are the spar chord forces P_1 and P_2 as well as the cover panel and web shear flows q_1 through q_4. The equilibrium conditions of the two spar chords, one rib chord and the vertical equilibrium conditions at the load locations Q_1, Q_2, will provide five equations in terms of the six unknown forces P_1, P_2, and q_1–q_4.

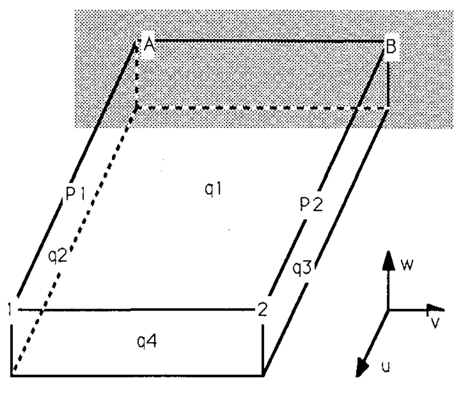

Fig. 1 Single-bay box beam.

Considering the free-body diagrams of the elements shown in Fig. 2, the following equilibrium equations may be readily derived:

$$
\begin{aligned}
P_1 + q_2 a - q_1 a &= 0 \\
P_2 + q_1 a + q_3 a &= 0 \\
q_1 b - q_4 b &= 0 \\
Q_1 - q_2 h - q_4 h &= 0 \\
Q_2 - q_3 h + q_4 h &= 0
\end{aligned}
\tag{1}
$$

Equation (1) may be written in a matrix form as $[A]\{F\} = [B]$, where the column vector $\{F\}$ contains the unknown forces P_1 and q_1–q_4 as shown below:

$$
\begin{bmatrix} -a & a & 0 & 0 & 1 \\ a & 0 & a & 0 & 0 \\ b & 0 & 0 & -b & 0 \\ 0 & -h & 0 & -h & 0 \\ 0 & 0 & -h & h & 0 \end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ P_1 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ -P_2 \\ 0 \\ -Q_1 \\ -Q_2 \end{Bmatrix}
\tag{2}
$$

Inverting the matrix $[A]$ and premultiplying it with $[B]$ will determine the internal forces $\{F\}$ as

$$
\begin{aligned}
q_1 &= -(Q_2/2h) - (P_2/2a) \\
q_2 &= (Q_2/2h) + (P_2/2a) + (Q_1/h) \\
q_3 &= (Q_2/2h) - (P_2/2a) \\
q_4 &= -(Q_2/2h) - (P_2/2a) \\
P_1 &= -(a/h)(Q_1 + Q_2) - P_2
\end{aligned}
\tag{3}
$$

The total strain energy stored in the box beam is now computed by summing up the strain energies stored in all elements of the box in terms of the axial forces P_1 and P_2, and the shear flows q_1–q_4 as

$$
\begin{aligned}
U = {} & (a/6A_f E)(2P_1^2 + 2P_2^2) + (1.3/Et)[(2/r)(q_1 a)^2 \\
& + (h/a)(q_2 a)^2 + (h/a)(q_3 a)^2 + (h/ar)(q_4 a)^2]
\end{aligned}
\tag{4}
$$

where $r = a/b$

In deriving this strain energy expression it should be noted that the strain energies of the top spar chords will be the same as those of the bottom spar chords. This is because equilibrium considerations will require the top and bottom spar-chord forces to be equal in magnitude but opposite in nature. Since

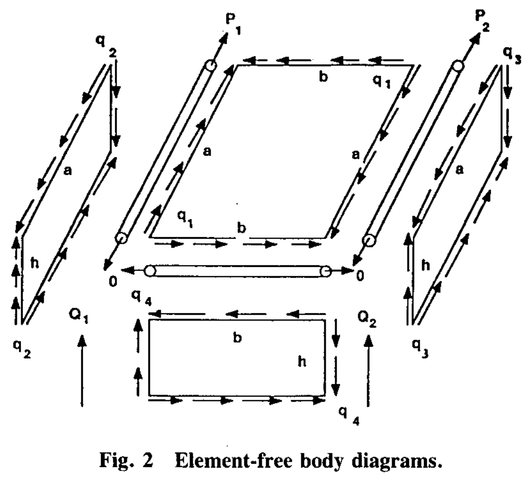

Fig. 2 Element-free body diagrams.

the shear flows q_1–q_4 and the axial force P_1 are already determined in Eq. (3) in terms of the redundant force P_2 and the applied loads Q_1 and Q_2, Eq. (4) may now be rewritten completely in terms of P_2 and Q_1 and Q_2. Taking the partial derivative of the total strain energy with respect to the redundant force P_2 and equating it to zero will provide the additional equation required to solve all the six unknown internal forces. These internal forces are now determined completely in terms of the applied external loads Q_1 and Q_2 as well as the geometric parameters of the box beam under consideration.

Since the internal forces are fully known, the total strain energy is now re-evaluated in terms of only the applied loads Q_1 and Q_2. The required lateral displacements at nodes 1 and 2 are then computed by taking the partial derivatives of the total strain energy with respect to the loads Q_1 and Q_2 as follows:

$$w_1 = \frac{\partial U}{\partial Q_1} \qquad w_2 = \frac{\partial U}{\partial Q_2} \tag{5}$$

The derivation of the equilibrium Eq. (1), rearranging these equations in the matrix form shown in Eq. (2), the solution of the system of equilibrium equations as given by Eq. (3), computation of the strain energy in Eq. (4), and finally taking partial derivatives of the strain energy to determine the lateral displacements, are all done systematically in a computer using the software package MACSYMA. At each stage, the solutions are available in the symbolic form for verification and/or numerical computation.

The numerical values of the desired lateral displacements can now be calculated by substituting for the physical dimensions, material parameters, and the actual external loads in the corresponding symbolic expressions. These steps are again carried out automatically by the computer. Using the SMT, the numerical results for internal forces and lateral displacements for several example problems are presented in Tables 1–6. In addition to the single-bay box beam, two types of 3-bay box beams, a 6-bay box beam and two cases of 12-bay box beams are analyzed. The results for each of these cases have been compared with the solutions obtained from a computer code.[5] The computer code was originally developed at the U.S. Army Ballistic Research Laboratory[4] and was recently reviewed, revised, and rewritten to generate reliable results for box beams with uniform cross section spar and rib chords and uniform thickness cover panels and shear webs.[5] Wherever possible, comparisons have also been made with alternate theoretical as well as available experimental test results.

The computer code results are based on Klein's method.[3] In this method a set of equilibrium equations are derived considering the free-body diagrams of the elements similar to the one shown in Fig. 2. For any given box beam, these equations will be identical to those derived for the energy method described earlier. Since each box beam problem is statically indeterminate, additional equations are obtained by considering the force-displacement relations of all the elements of the box structure. Thus, the basic assumptions are the same in Klein's method as well as in the energy approach.

Table 1 Results for single-bay box beam

Internal forces and displacements	Energy	CC[a]	CC[b]	SMT
	$Q1 = 1, Q2 = 0$			
p_1	−9.4795	−9.8536		−9.4795
p_2	−6.2315	−5.8574		−6.2315
q_1a	3.1157	2.9287		3.1157
q_2a	12.5952	12.7822		12.5952
q_3a	3.1157	2.9287		3.1157
q_4a	3.1157	2.9287		3.1157
$w1$	4814.6800	3907.5900	4845.90	4814.6800
$w2$	2896.4700	2080.0200	2927.10	2896.4700
	$Q1 = Q2 = 1$			
p_1	−15.7109	−15.7109		−15.7109
p_2	−15.7109	−15.7109		−15.7109
q_1a	0.0000	0.0000		0.0000
q_2a	15.7109	15.7109		15.7109
q_3a	15.7109	15.7109		15.7109
q_4a	0.0000	0.0000		0.0000
$w1$	7711.1400	5987.6000	7772.90	7711.1400[c]
$w2$	7711.1400	5987.6000	7772.90	7711.1400[c]

[a]Computer code, seven elements.
[b]Computer code, 13 elements.
[c]Strength of materials solution with shear correction, 7740 (Ref. 8).
a = 400 in., b = 254.6, h = 25.46, t = 0.05, A_f = 3.1825 square in., E = 1 psi, ν = 0.3; Q_i, P_i, q_ia is in pounds and w_i is in inches.

Table 2 Results for three-bay box beam (Fig. 3)

	$Q1 = 1$			$Q1 = Q4 = 1$		
Displacements	CC[a]	CC[b]	SMT	CC[a]	CC[b]	SMT
$w1$	3218.50	3615.90	3582.79	3243.90	3750.50	3699.85
$w2$	1590.00	1879.10	1810.93	2175.20	2653.80	2541.73
$w3$	585.40	774.76	730.80	2175.20	2653.80	2541.85
$w4$	25.26	134.64	117.07	3243.90	3750.50	3699.85

[a]Computer code, 19 elements.
[b]Computer code, 35 elements.
a = 400 in., b = 254.6, h = 25.46, t = 0.05, A_f = 3.1825 square in., E = 1 psi, ν = 0.3; Q_i is in pounds and w_i is in inches.

Table 3 Results for three-bay box beam (Fig. 4)

	$Q1 = 1$			$Q1 = Q4 = 1$		
Displacements	CC[a]	CC[b]	SMT	CC[a]	CC[b]	SMT
$w1$	2691.60	2732.80	2600.00	4644.40	4724.30	4464.00
$w2$	1494.30	1522.30	1433.75	2527.00	2580.30	2407.11
$w3$	521.00	536.17	491.69	836.30	863.00	776.89
$w4$	1952.80	1991.50	1863.52	4644.40	4724.30	4464.00
$w5$	1032.70	1058.00	973.36	2527.00	2580.30	2407.11
$w6$	315.33	326.83	285.20	836.30	863.00	776.90

[a]Computer code, 19 elements.
[b]Computer code, 37 elements.
$a = b$ = 40 in., h = 10, t = 0.05, A_{eff} = 3 square in., E = 1 psi, ν = 0.3; Q_i is in pounds and w_i is in inches.

Table 4 Results for six-bay box beam

	$Q1 = 1$				$Q1 = Q7 = 1$			
	CC[a]	CC[b]	SMT	FEM[c]	CC[a]	CC[b]	SMT	FEM[c]
$w1$	138.13	138.96	132.42	131.16	132.14	133.24	124.17	122.06
$w2$	61.63	62.05	58.98	58.87	57.28	57.28	52.99	52.82
$w3$	74.61	75.42	69.92	68.46	106.38	106.38	96.85	94.03
$w4$	36.30	36.72	34.14	33.50	51.79	51.79	47.28	45.76
$w5$	30.33	30.96	26.93	25.39	106.44	106.44	96.85	93.52
$w6$	14.75	15.07	13.14	11.26	51.81	51.81	47.28	42.88
$w7$	−5.99	−5.71	−8.26	−9.15	133.22	133.22	124.17	122.25
$w8$	−4.88	−4.76	−6.00	−4.90	57.28	57.28	52.99	56.01

[a]Computer code, 35 elements.
[b]Computer code, 67 elements.
[c]Finite element method—ANSYS, 70 nodes and 124 elements.
$a = 40$ in., $b = 30$, $h = 25$, $t = 0.05$, $A_{eff} = 10$ square in., $E = 1$ psi, $\nu = 0.3$; Q_i is in pounds and w_i is in inches.

Table 5 Results for 12-bay box beam

	$Q1 = 1$				$Q1 = Q13 = 1$			
	CC[a]	CC[b]	SMT	FEM[c]	CC[a]	CC[b]	SMT	FEM[c]
$w1$	226.78	227.62	214.62	213.47	211.83	212.72	195.96	198.64
$w2$	135.46	136.06	127.30	127.72	121.11	122.11	111.02	114.30
$w3$	59.18	59.53	55.18	56.06	50.55	50.90	45.50	47.82
$w4$	146.02	147.12	135.72	136.86	178.24	180.03	162.16	167.02
$w5$	92.86	93.61	86.08	87.85	112.49	113.69	101.94	106.64
$w6$	40.89	41.28	37.60	39.23	48.72	49.30	43.64	47.08
$w7$	83.84	84.96	75.80	79.25	167.74	169.97	151.59	158.55
$w8$	54.28	55.03	49.02	52.16	108.60	110.09	98.04	104.34
$w9$	23.90	24.24	21.36	24.16	47.81	48.49	42.73	48.33
$w10$	32.10	32.86	26.44	30.09	178.04	179.97	162.16	166.89
$w11$	19.56	20.05	15.85	18.77	112.38	113.66	101.94	106.62
$w12$	7.80	8.01	6.05	7.85	48.69	49.28	43.64	47.06
$w13$	−14.85	−14.82	−18.66	−14.88	211.91	212.74	195.96	198.49
$w14$	−13.88	−13.89	−16.28	−13.42	121.56	122.12	111.02	114.29
$w15$	−8.61	−8.61	−9.69	−8.24	50.56	50.91	45.50	47.85

[a]Computer code, 67 elements.
[b]Computer code, 130 elements.
[c]Finite element method—ANSYS, 40 nodes and 132 elements.
$a = 40$ in., $b = 30$, $h = 25$, $t = 0.05$, $A_{eff} = 10$ square in., $E = 1$ psi, $\nu = 0.3$; Q_i is in pounds and w_i is in inches.

Table 6 Comparison of results for Rattinger's wing lateral displacements at nodes (in inches)

Node	Ref. 7	SMT	Test
1	0.1771	0.1864	0.1824
2	0.0890	0.1015	0.1010
3	0.0227	0.0316	0.0324
4	0.1671	0.1728	0.1687
5	0.0852	0.0940	0.0940
6	0.0240	0.0284	0.0298
7	0.1595	0.1603	0.1571
8	0.0810	0.0867	0.0846
9	0.0229	0.0253	0.0269
10	0.1506	0.1492	0.1449
11	0.0748	0.0796	0.0772
12	0.0196	0.0222	0.0234
13	0.1417	0.1390	0.1341
14	0.0672	0.0726	0.0708
15	0.0142	0.0191	0.0183

100-lb load applied at node 1; see Fig. 7 for details.

In both cases, the formulation does not completely satisfy compatibility conditions continuously along the shear panel and spar chord connections.

Discussion of Results

Several example problems that are considered in this article are illustrated in Figs. 1–7. A single-bay box beam shown in Fig. 1 has six internal forces P_1, P_2, and q_1–q_4. The results for the internal forces as well as the lateral displacements at nodes 1 and 2 are given in Table 1 for symmetric and unsymmetric loadings. Current results using the SMT are presented along with those of the energy method. The results of the energy method have been computed independently by generating the equilibrium equations by hand and then applying minimization principles. CC stands for the solutions obtained from the computer code.[5] It is clear that the results of the energy method and those of the symbolic manipulation technique are identical. Since the computer code is based on the Klein's method[3] (which is not fully capable of correcting the displacements for the shear effects) the lateral displacements of the computer code are in error. By increasing the number of nodes and dealing with larger number of shear web elements it is possible to improve the results of the computer code as shown in Table 1. Comparisons have also been made in Table 1 with some results available from the literature using the modified strength-of-materials theory to include shear corrections.

Two cases of three-bay box beams are shown in Figs. 3 and 4. The dimensions of the box, the areas of cross sections of spar and rib chords, and the material properties are listed under each example problem in Tables 1–6. In all the cases considered in this article, the elastic modulus has to be taken as unity to minimize the round-off errors while using the computer code and therefore, the same value has been used throughout for comparison. The actual diplacements are obtained by dividing the values given in this article by the true value of the elastic modulus. The total number of unknown internal forces for the box beam shown in Fig. 3 are 16, whereas for the box beam shown in Fig. 4 the number will be 18. The nodal lateral displacements for these two cases are presented in Tables 2 and 3 with those of the computer code solutions. Alternate theoretical results using the energy method could not be generated due to the complex nature of these problems.

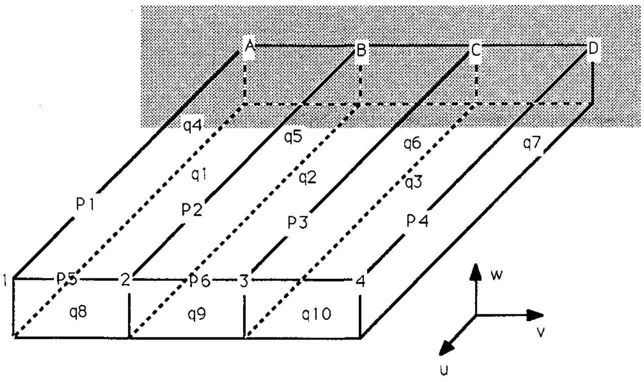

Fig. 3 Three-bay box beam, case I.

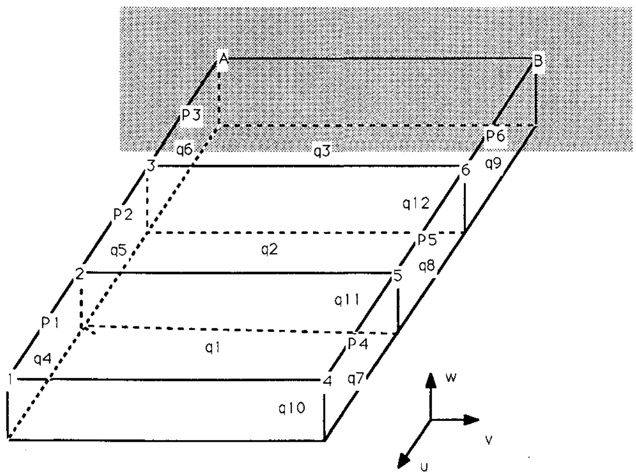

Fig. 4 Three-bay box beam, case II.

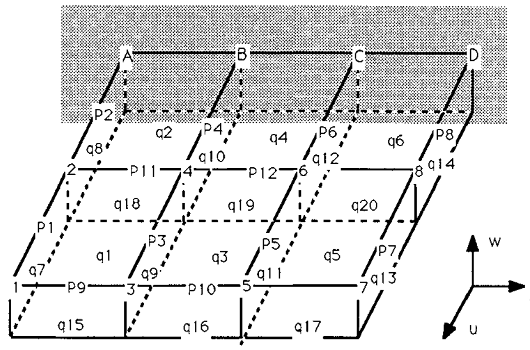

Fig. 5 Six-bay box beam.

Numerical results for the lateral displacements are presented in Table 4 for a six-bay box beam. The results of SMT have again been compared with those generated using the computer code. In this case the total number of unknown internal forces are 32. For each case, two sets of computer code solutions are presented. The first set corresponds to the basic model of the box beam under consideration and the second set corresponds to an improved model to incorporate additional nodes and elements to provide for better shear corrections to the lateral displacements. Comparisons with finite-element solutions indicate good agreement with the SMT results as shown in Table 4.

In Tables 5 and 6, results corresponding to two cases of 12-bay box beams are tabulated considering both symmetric as well as unsymmetric loadings. In both cases there are 63 unknown internal forces. In Table 5 the results of SMT are compared with two sets of computer code solutions as well as finite-element results. Here again, the SMT results are found to be in good agreement with finite element results. In Table 6 SMT results are presented for a Rattinger's wing and

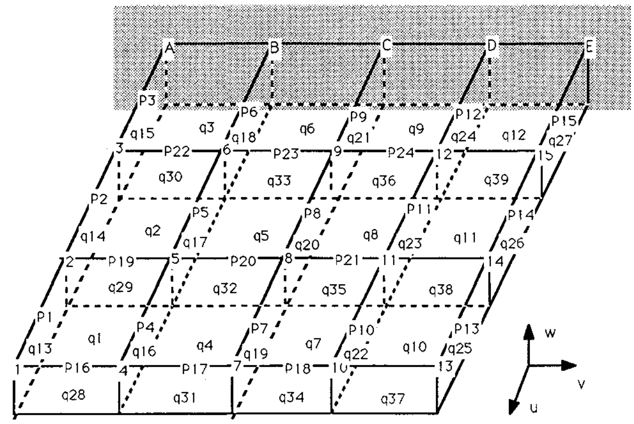

Fig. 6 Twelve-bay box beam.

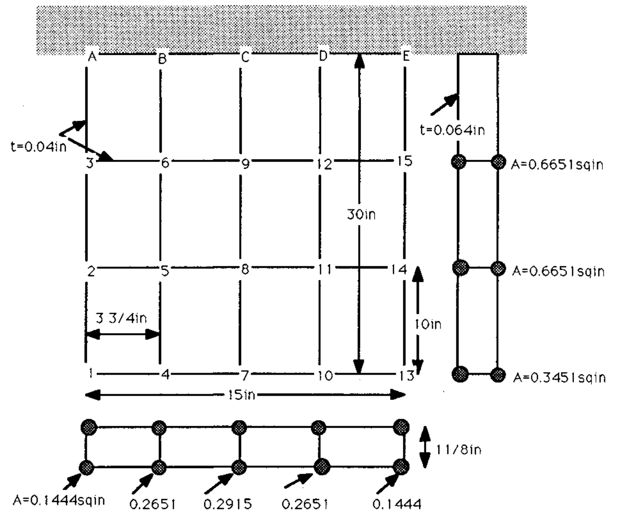

Fig. 7 Rattinger's wing.

these results have been compared with both experimental as well as alternate theoretical results. The alternate theoretical results presented here are based on the application of the Rayleigh-Ritz method with the aid of Legendre Polynomials as given in Ref. 7.

A careful review of the results presented in Tables 1–6 indicates that the computer code which is based on the Klein's method, may be used for the analysis of regular, symmetrical box beams built of rectangular boxes. It is assumed that all the box structures considered in this article are composed of uniform, nontapered axial elements (spar and rib chords) taking only axial loads and constant thickness sheets (cover panels, spar and rib webs), taking only shear loads in the form of constant shear flows. In many cases the results of the computer code can be improved by considering additional nodes, thereby increasing the number of spar and rib web elements. This procedure will provide for additional shear corrections to the lateral displacements.

The choice of the appropriate type of finite element for chords and webs is very important when using finite-element codes for comparison. In the case of ANSYS, a three-dimensional spar element (STIF 8) has been chosen to represent the spar and rib chords. This element has two nodes with three DOF for each node. A quadrilateral shell element (STIF 63) consisting of four nodes with six DOF at each node was used for cover panels, spar and rib webs.

A new method based on the energy principles has been developed and applied to the analysis of several box beam problems. Numerical results presented in Tables 1–6 indicate very good agreements with alternate theoretical and experimental test results as well as finite-element solutions. Although only symmetrical box beams have been considered in this article, the method could be extended to deal with unsymmetrical configurations. The method is also capable of handling unsymmetrical box beams with variable thickness cover panels, variable spar and rib chord areas, and unsymmetrical top and bottom surfaces. It is also easy to model damages in this method. Work is under progress to extend this method to the analysis of real aircraft wings which often have the variations described above.

Acknowledgments

Financial support was provided to M. Sathyamoorthy under an IPA Contract. The authors are thankful to Donald F. Haskell, Chief of the Air System Branch/VLD, U.S. Army Ballistic Research Laboratory at Aberdeen Proving Ground for his support and encouragement throughout this research.

References

[1]Fredriksson, B., "Advanced Numerical Methods for Analysis and Design of Aircraft Structures," *International Journal of Vehicle Design*, Vol. 7, Nos. 3 and 4, 1986, pp. 306–336.

[2]Levy, S., "Computation of Influence Coefficients for Aircraft Structures with Discontinuities and Sweepback," *Journal of the Aeronautical Sciences*, Vol. 14, Aug. 1947, pp. 547–560.

[3]Klein, B., "A Simple Method of Matrix Structural Analysis," *Journal of the Aeronautical Sciences*, Vol. 24, 1957, pp. 39–46.

[4]Oehrli, R., "Wing Aeroelastic Analysis: Maneuvering Target," Vol. III, U.S. Army Ballistic Research Lab., Aug. 1988, pp. 1–125.

[5]Sathyamoorthy, M., and Klenke, T., "Evaluation and Application of Klein's Method for Automated Damaged Wing Structural Analysis," Rept. TCN 88234, U.S. Army Research, Aug. 1988, pp. 1–120.

[6]MACSYMA Reference Manual, MIT Lab. for Computer Science, Vols. I and II, Jan. 1983.

[7]Greenspon, J. E., and Singer, E. G., "Rayleigh-Ritz Procedures for Predicting Static Deflections, Natural Frequencies and Mode Shapes of Low Aspect Ratio Wings and Control Surfaces," Rept. AB-88-1, U.S. Army Ballistic Research Lab., March 1988, pp. 1–57.

[8]Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," *Journal of the Aeronautical Sciences*, Vol. 23, June 1956, pp. 805–823.